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MEASUREMENTS OF THE RADIAL AND TANGENTIAL EDDY DIFFUSIVITIES OF HEAT AND MASS IN TURBULENT FLOW IN A PLAIN TUBE

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Abstract—An experimental study was made of the variation of the radial and tangential eddy diffusivities of heat and mass in a fully developed turbulent flow in a plain tube. The results for heat cover the Reynolds number range from 10 000 to 160 000 and for mass from 6000 to 170 000. In both cases air was the working fluid with a corresponding Prandtl number 0·71 and with nitrous oxide injection a Schmidt number 0·77.

The results for mass transfer were obtained by diffusing nitrous oxide from a ring source mounted flush in the wall of the tube for the radial eddy diffusivity and by diffusion from a small patch source at the tube wall for the tangential eddy diffusivity. The results for heat transfer were obtained using an electrically heated wall ring source or a similar wall patch source for the radial and tangential eddy diffusivities respectively.

It was found that the radial eddy diffusivities of both heat and mass could be expressed as a simple ratio of the radial eddy diffusivity of momentum for all the Reynolds numbers considered. This ratio is a simple function of the non-dimensional radius and it has a value of about two near the wall but varies smoothly to unity at the centre of the tube.

Similarly it was found that each of the tangential eddy diffusivities could be expressed as a simple ratio of its corresponding radial eddy diffusivity; this ratio is also a simple function of non-dimensional radius. Its value is very high at the wall but seems about unity over the greater part of the tube.

All the experimental results presented were processed numerically by computer. An appendix deals with the methods used.

NOMENCLATURE

- b, constant coefficient;
- c, concentration;
- c_b, bulk concentration;
- c_p, specific heat at constant pressure;
- D, molecular mass diffusivity;
- g, gravitational constant;
- Gr, Grashof number $\dfrac{8g\beta r_0^3(t_0 - t)}{\nu^2}$;
- k, molecular conductivity;
- l, mixing length;
- L_n, layering number $\dfrac{u_{max}}{\left[\dfrac{gq}{2r_0}\dfrac{\Delta\rho}{\rho}\right]^{\frac{1}{3}}}$;
- n, constant coefficient;
- Pr, Prandtl number $\mu c_p/k$;
- q, diffusion rate;
- r, radius;
- r^+, ru^*/ν;
- r_0, radius of tube;
- r_0^+, r_0u^*/ν;
- Re, Reynolds number, $2\,u_b\,r_0/\nu$;
- Sc, Schmidt number ν/D;
- t, temperature;
- t_b, bulk temperature;
- u, velocity of fluid;
- u^+, u/u^*;
- u^*, friction velocity $\sqrt{(\tau_0/\rho)}$;
- u_b, bulk velocity;
- u', fluctuating velocity in axial direction;
- v', fluctuating velocity in radial direction;

w', fluctuating velocity in tangential direction;
x, axial distance along tube;
x^+, x/r_0;
y, radial distance from wall;
y^+, yu^*/ν;
z, non-dimensional radius r/r_0.

Greek symbols
α, molecular thermal diffusivity;
β, volumetric expansion coefficient;
ε, eddy diffusivity;
θ, non-dimensional concentration or temperature c/c_b or t/t_b;
ν, kinematic viscosity;
ξ, ratio of $\varepsilon_{h,r}/\varepsilon_m$ or $\varepsilon_{d,r}/\varepsilon_m$;
ρ, density;
τ, shear stress;
τ_0, shear stress at the wall;
ψ, ratio of $\varepsilon_{h,\omega}/\varepsilon_{h,r}$ or $\varepsilon_{d,\omega}/\varepsilon_{d,r}$;
ω, angular co-ordinate.

Subscripts
b, bulk;
d, mass;
h, heat;
i, intercept;
l, at edge of sub-layer;
m, momentum;
max, maximum;
0, wall;
r, radial;
ω, tangential.

INTRODUCTION

MOST of the existing theoretical and experimental work on heat and mass transfer in a turbulent flow in a plain tube has considered only axisymmetric cases. The non-axisymmetric cases, which predominate in practice, have received much less attention and that only recently. The solution of such problems requires a knowledge of the eddy diffusivity in the tangential direction as well as in the radial direction. Recent work on this topic has been much concerned with establishing the tangential eddy diffusivity by relating it through some ratio to the radial eddy diffusivity. This is a logical extension of the method of solution of the axisymmetric cases where the radial eddy diffusivity of heat or mass is obtained from that of momentum by some modification of Reynolds analogy.

Sparrow and Lin [1] and Reynolds [2] considered heat transfer in a fully developed turbulent flow in a plain tube with circumferential variation of wall heat flux or temperature. Both assumed that the tangential eddy diffusivity of heat, $\varepsilon_{h,\omega}$, equalled the radial $\varepsilon_{h,r}$, at the same point. But later experimental evidence for an asymmetrically heated tube, Sparrow and Black [3], suggested that, in the sublayer region close to the wall the ratio might be about ten. Quarmby and Anand [4] gave theory and experiment for non-axisymmetric mass transfer using nitrous oxide diffusing into air. They concluded the ratio was unity; that is, the tangential eddy diffusivity of mass, $\varepsilon_{d,\omega}$, equalled the radial, $\varepsilon_{d,r}$, at the same point in the flow.

Recent calculations by Quarmby [5] have shown that for Prandtl number, Pr, or Schmidt number, Sc, of about 0·7 or less and unless the Reynolds number, Re, is high the agreement between the theoretically predicted concentration profiles and experiment in [4] would have been as good within the limits of experimental error had the theory assumed a ratio of $\varepsilon_{d,\omega}/\varepsilon_{d,r}$ which varied smoothly from two at the wall to unity at the centre. Also, most surprisingly, if the ratio was unity in the main stream it could be infinite, even, in the sub-layer without altering the prediction for $Pr \leqslant 0{\cdot}7$ to any measurable extent. Though, this is not to say that an accurate knowledge of the radial diffusivity ratio is unimportant there. Reference [4] did not investigate Reynolds numbers below about 20000 and thus they were not able to make measurements in the sub-layer. The sub-layer region is most important for heat or mass transfer at high Prandtl or Schmidt numbers. Certain hot-wire measurements of the fluctuating velocities, Laufer [6] and Sandborn [7], suggest that the tangential component is greater than the radial near the wall. Clearly it is of great interest to obtain measurements of the tangential eddy diffusivity in the sub-layer. From Laufer's measurements we would expect the ratios $\varepsilon_{d,\omega}/\varepsilon_{d,r}$ and $\varepsilon_{h,\omega}/\varepsilon_{h,r}$ to be greater than unity there. In the present work low Reynolds numbers are investigated and some attempt made to obtain measurements in the sub-layer.

The diffusion of nitrous oxide gas in air has been used as a model to investigate heat transfer situations of interest. The analogy between heat and mass transfer is probably quite valid since both are scalar quantities and conclusions drawn from the mass transfer situation should be relevant to the heat transfer situation. On the other hand, use of the results of [4] for calculations of non-axisymmetric heat transfer is not absolutely certain to be correct; especially, in the light of the results of Sparrow and Black. Although experiments with heat transfer are more difficult than with mass transfer, because of the imprecision of the boundary conditions, it is clearly better to have results for heat transfer, if possible, which parallel those of [4] for mass transfer. The present work attempts this.

FORMULATION OF THE GOVERNING EQUATIONS

The temperature, t, at any point, x, r, ω in a fully developed turbulent flow in a plain tube is given by

$$u\frac{\partial t}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\alpha + \varepsilon_{h,r})\frac{\partial t}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \omega}\left[(\alpha + \varepsilon_{h,\omega})\frac{\partial t}{\partial \omega}\right]. \tag{1}$$

We assume steady state conditions and constant fluid properties and neglect axial diffusion. This equation holds equally for mass transfer if in the place of temperature, t, we put the concentration c, and replace α by D and and $\varepsilon_{h,r}$, $\varepsilon_{h,\omega}$ by $\varepsilon_{d,r}$ and $\varepsilon_{d,\omega}$ respectively. Using non-dimensional variables, equation (1) becomes

$$u^+\frac{\partial \theta}{\partial x^+} = \frac{r_0^+}{r^+}\frac{\partial}{\partial r^+}\left[r^+\left(\frac{1}{Pr} + \frac{\varepsilon_{h,r}}{\nu}\right)\frac{\partial \theta}{\partial r^+}\right] + \frac{r_0^+}{r^{+2}}\frac{\partial}{\partial \omega}\left[\left(\frac{1}{Pr} + \frac{\varepsilon_{h,\omega}}{\nu}\right)\frac{\partial \theta}{\partial \omega}\right]. \tag{2}$$

The basic problem is to evaluate $\varepsilon_{h,\omega}$ if we have experimental measurements of θ as a function of x^+, r^+ and ω. Clearly, we would need to know u^+ and $\varepsilon_{h,r}$ for each value of r^+. There are many theories to describe the $u^+ \sim y^+$ profile in a plain tube. Measurements of $\varepsilon_{h,r}$ may be obtained from an axisymmetric experimental situation. In that case, equation (2) becomes

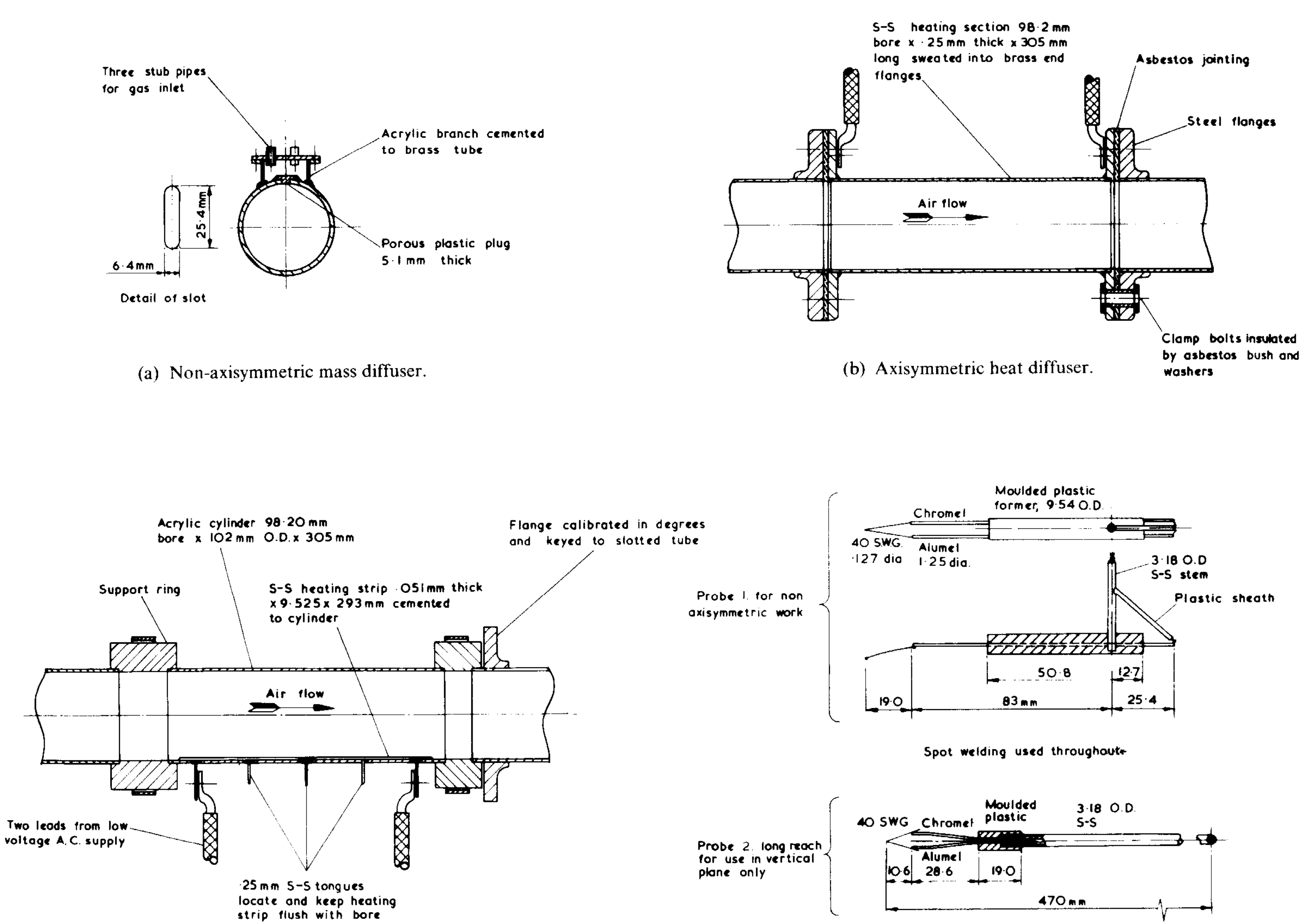
$$u^+\frac{\partial \theta}{\partial x^+} = \frac{r_0^+}{r^+}\frac{\partial}{\partial r^+}\left[r^+\left(\frac{1}{Pr} + \frac{\varepsilon_{h,r}}{\nu}\right)\frac{\partial \theta}{\partial r^+}\right] \tag{3}$$

which may be re-arranged as

$$\frac{\varepsilon_{h,r}}{\nu} = \frac{\int_0^{r^+} u^+ r^+ \frac{\partial \theta}{\partial x^+}\,dr^+}{r_0^+ \frac{\partial \theta}{\partial r^+} r^+} - \frac{1}{Pr}. \tag{4}$$

Similarly, to evaluate, $\varepsilon_{h,\omega}$ equation (2) may be written

$$\frac{\varepsilon_{h,\omega}}{\nu} = \frac{r^{+2}\left(u^+\frac{\partial \theta}{\partial x^+} - \frac{r_0^+}{r^+}\frac{\partial}{\partial r^+}\left[r^+\left(\frac{1}{Pr} + \frac{\varepsilon_{h,r}}{\nu}\right)\frac{\partial \theta}{\partial r^+}\right]\right)}{r_0^+\frac{\partial^2 \theta}{\partial \omega^2}} - \frac{1}{Pr} \tag{5}$$



(a) Non-axisymmetric mass diffuser.

(b) Axisymmetric heat diffuser.

(c) Non-axisymmetric heat diffuser.

(d) Temperature probes.

FIG. 1. Experimental apparatus.

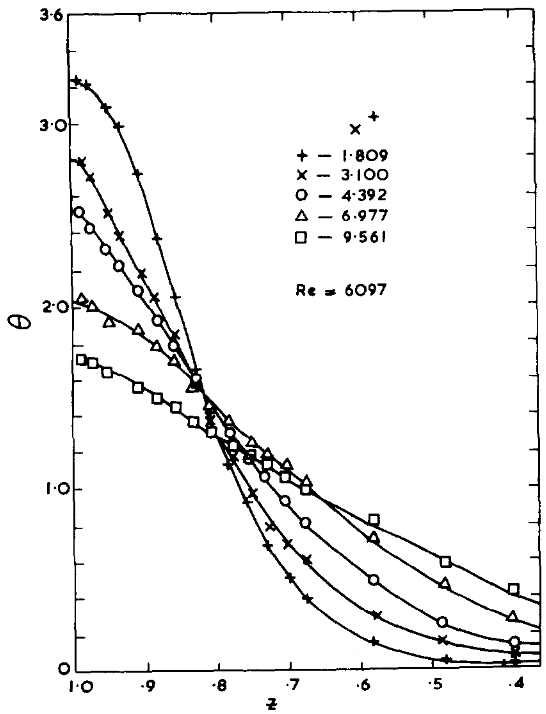


FIG. 2. Axisymmetric mass transfer, concentration profiles.

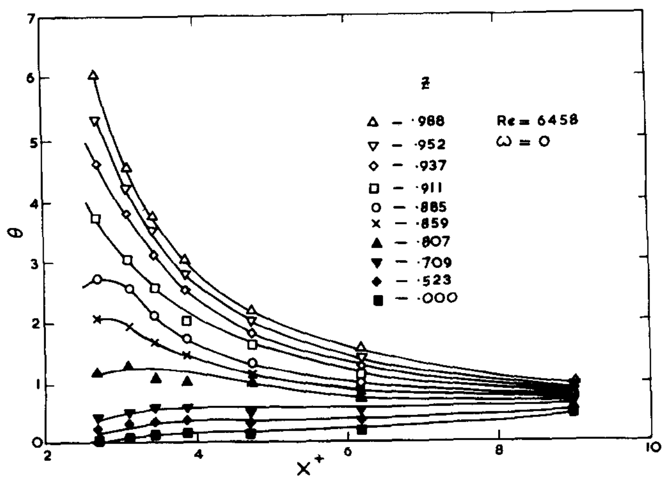


FIG. 3. Non-axisymmetric mass transfer, concentration profiles.

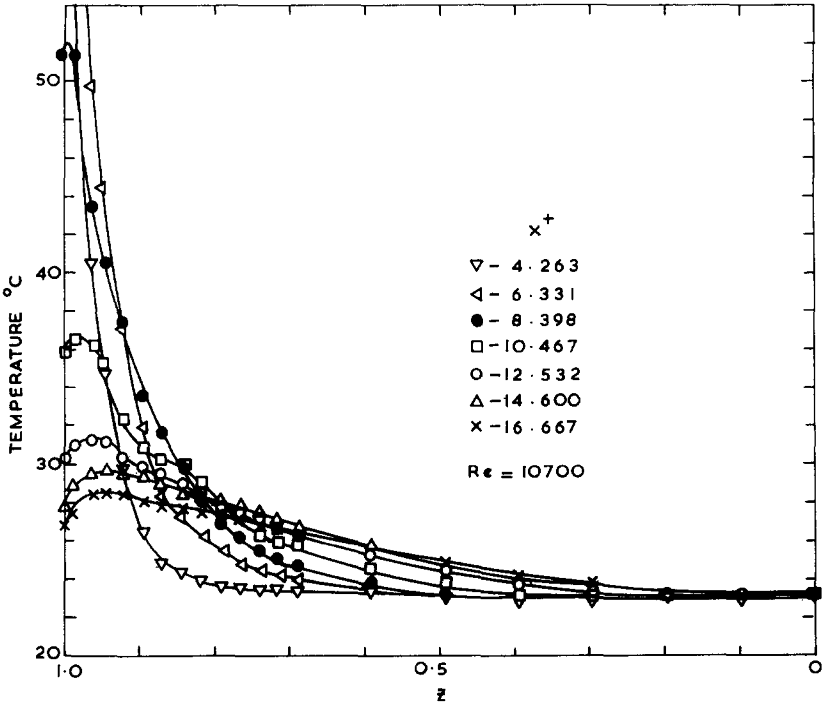


Fig. 4. Axisymmetric heat transfer, temperature profiles.

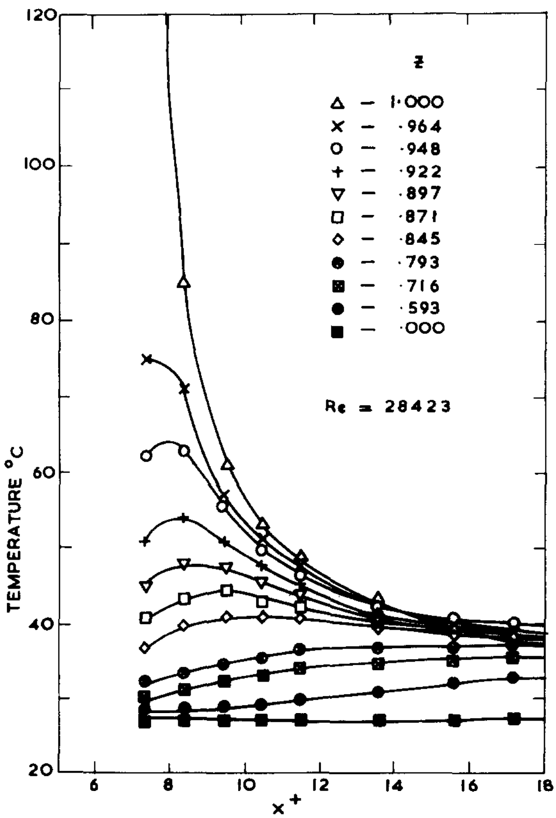


Fig. 5. Axisymmetric heat transfer, temperature profiles.

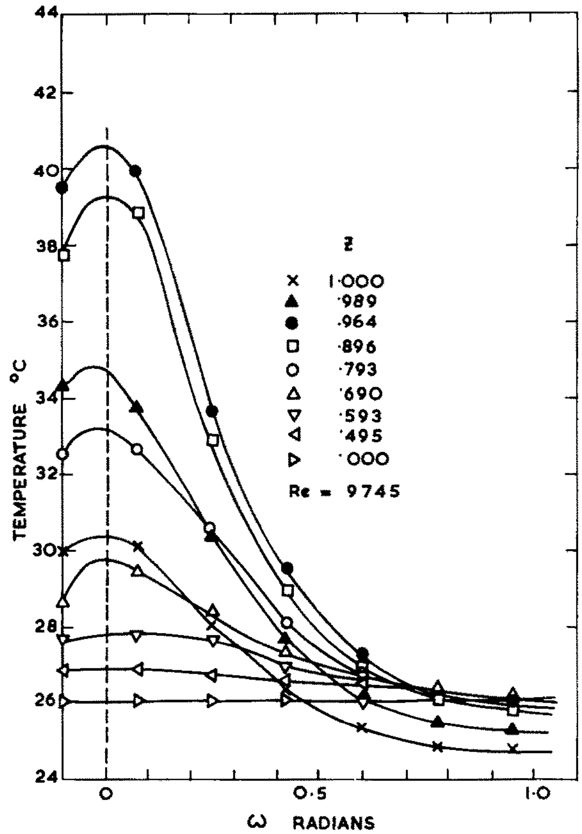


FIG. 6. Non-axisymmetric heat transfer, temperature profiles.

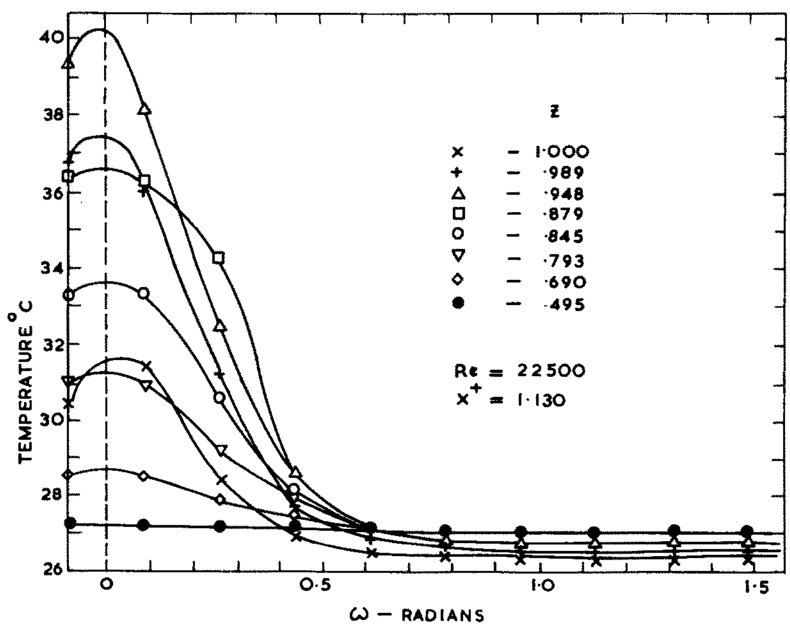


FIG. 7. Non-axisymmetric heat transfer temperature profiles.

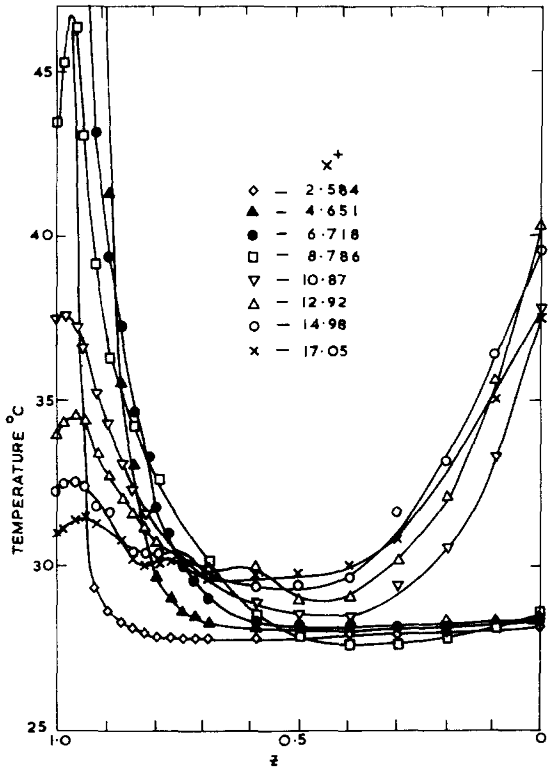


FIG. 8. Axisymmetric heat transfer, effect of natural convection.

where it is recognised that $\varepsilon_{h,\omega}$ is not a function of ω. To evaluate $\varepsilon_{h,r}$, we need the first derivative of the temperature profile in the radial and axial direction at each point in the flow; whilst evaluating $\varepsilon_{h,\omega}$ requires those and, also, the second derivatives.

Evaluating derivatives from experimental data is not easy. It has often been done graphically. That is a rather dubious process. More recently, Sleicher [8] and [4] used simple numerical methods. It is clear from equations (4) and (5) that the value of the results for $\varepsilon_{h,r}$ and $\varepsilon_{h,\omega}$ will depend very much on the accuracy of determining the derivatives. In this work a considerable amount of effort was devoted to devising reliable numerical methods. These are described in the Appendix.

EXPERIMENTAL INVESTIGATION

The basic apparatus was that used by [4]. It consisted of a plain brass tube 98·3 mm (3·870 in.) bore and 13 m (42·75 ft) long of which 5·8 m (19·25 ft) was developing section. For low Reynolds numbers the mass flow was determined by traversing the velocity profile in a smaller tube, about 42·9 mm bore, placed on the exit of the working section and the small mass flows of nitrous oxide by a fine capillary flow meter.

The small patch source for non-axisymmetric mass transfer is shown in Fig. 1a. The axisymmetric ring source for heat transfer was made by forming stainless steel sheet 0·254 mm (0·010 in.) thick to the exact bore of the tube, Fig. 1b. For non-axisymmetric heating a narrow strip of the same stainless steel 0·051 mm (0·002 in.) thick was set in a plastic tube, bored to the same dimension as the brass tube. Stainless steel was chosen for its high electrical resistivity. Heat fluxes were up to about 6·9 kW/m^2 and 2·7 kW/m^2 and surface temperature rises of the heaters 150°C and 70°C in the symmetric and non-symmetric cases respectively. The temperature probe is shown in Fig. 1d.

Local mass concentration measurements were obtained by drawing samples of the air and nitrous oxide mixture through a fine capillary tube probe into an infra-red gas analyser. The details of this system are given in [4].

DESCRIPTION OF THE VELOCITY AND EDDY DIFFUSIVITY OF MOMENTUM

It is very clear that the value of any results for the ratio between the eddy diffusivities of heat or mass and the eddy diffusivity of momentum, determined in the experiments described above, will depend greatly on the accuracy of the description of the latter.

Measurements of ε_m have been made by many researchers. It can be determined from the velocity profile and wall shear measurements by

$$\frac{\varepsilon_m}{\nu} = \left(1 - \frac{y^+}{y_0^+}\right) \bigg/ \frac{du^+}{dy^+} - 1 . \tag{6}$$

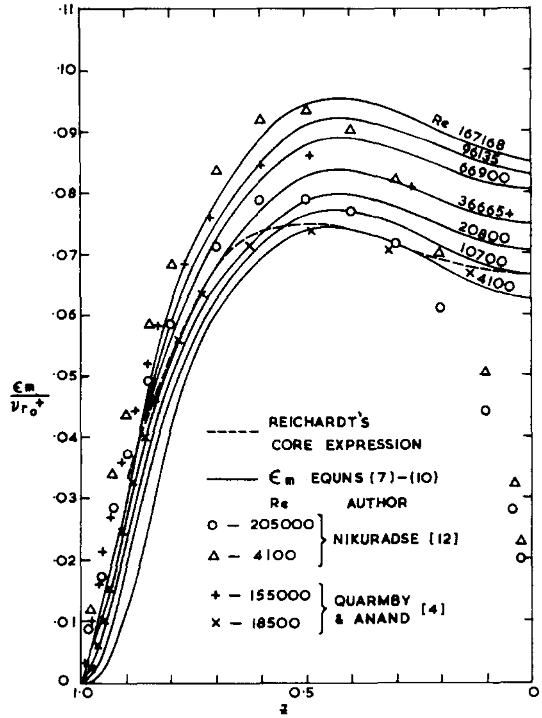


FIG. 9. Eddy diffusivity of momentum, comparison of theory with experiment.

Figure 9 shows the results of [4], Nikuradse [12] and Reichardt [13]. It has often been assumed, [13] and many others, that such results can be described by a single curve and there is no effect of Reynolds number. In the half of the flow nearest the wall this is a reasonable assumption but at the centre the results show some divergence. It is very difficult to evaluate ε_m near the centre from equation (6). In fact, a 1 per cent error limit on the determination of u^+ gives a spread somewhat greater than ± 60 per cent in the result for ε_m in that region.

Figure 9 shows the differences that exist between the various experimental results. Nikuradse shows $\varepsilon_m/\nu r_0^+$ decreasing with Reynolds number, Quarmby and Anand show it increasing with Reynolds number and Reichardt shows no Reynolds number dependence at all. It is clear that some considerations other than a subjective assessment of which is the 'best' of these results are needed to establish the correct eddy diffusivity of momentum. One would be whether or not the description of ε_m chosen gave an acceptable velocity profile when integrated, equation (6). Reference [4] used a description of ε_m based on the work of Van Driest [14] and [13]. This description had some Reynolds number dependence and gave a family of curves for $\varepsilon_m/\nu r_0^+$. However when integrated to derive $u^+ \sim y^+$ from equation (6) it did not give the best description of the velocity profile. Reference [4] instead used the description of the velocity profile proposed by Quarmby [15] based on Von Karman's similarity hypothesis and Deissler's [16] sub-layer profile. One drawback of the Von Karman hypothesis is that it predicts ε_m to be zero at the centre line. Thus in [4], the descriptions of u^+, y^+ and ε_m were formulated separately to give the best agreement with experimental results but they were not completely compatible according to equation (6).

The prediction of ε_m from the Von Karman hypothesis as modified by [15] is good except in the centre. As stated, the measurements of ε_m at the centre can be somewhat unreliable. Accordingly it was decided to use the results of [15] to describe the ε_m profile in the region from the wall up to a certain value of the radius and to assume that from there to the centre ε_m was given by a Reichardt type Reynolds number dependent expression which was a good fit to the available data. On integrating equation (6) using this description of ε_m, it was found that the resulting $u^+ \sim y^+$ gave as good agreement with experiment for the plain tube as did the results of [14] and, in fact, predicted the Reynolds number effect on the $u^+ \sim y^+$ plot equally well.

It is clear from Reichardt's work that he considered that the variation of $\varepsilon_m/\nu r_0^+$ with z had a Reynolds number dependence but chose an expression which did not have such a dependence for simplicity. Gill and Scher [17] modified the Prandtl mixing length which they considered was a function of Reynolds number and obtained a variation of $\varepsilon_m/\nu r_0^+$ with Reynolds number. Most recently, Travis *et al.* [18] listed the constraints which must act on an acceptable $u^+ \sim y^+$ description. They derived a modified Reichardt type expression for $\varepsilon_m/\nu r_0^+$ with four Reynolds number dependent variables which satisfies the majority of these constraints.

Accordingly, it seems clear that it was incorrect to assume that $\varepsilon_m/\nu r_0^+$ is not a function of Reynolds number. Previous expressions for the ratio of $\varepsilon_{h,r}/\varepsilon_m$ and $\varepsilon_{d,r}/_m$ may be, correspondingly, in error.

it may be approximated by an equation of the form

$$\frac{\varepsilon_m}{\nu} = r_0^+ \sum_{j=0}^{4} b_j z^j$$

where

$$b_j = \sum_{i=0}^{4} a_{ij} \left(\frac{Re}{10^5}\right)^j \qquad (10)$$

The coefficients, a_{ij}, in equation (10) are given in Table (1).

Table 1. *Coefficients in equation* (10)

i \ j	0	1	2	3	4
0	0·063434	−0·082104	0·651923	−1·137887	0·498520
1	0·047545	0·439134	−2·862303	5·244256	−2·851530
2	0·046523	−0·980515	6·412239	−11·870411	6·458515
3	0·026302	0·871095	−5·680879	10·559449	−5·756526
4	−0·006360	−0·255320	1·662109	−3·0928523	1·687192

The variations of ε_m used in this work is shown in Fig. 9. When compared with the much more detailed analysis of Travis *et al.*, the agreement is satisfactory. This variation was described by the following equations.

From [15] and [16] for $0 < y^+ < y_l^+$

$$\frac{\varepsilon_m}{\nu} = n^2 u^+ y^+ \left[1 - \exp(-n^2 u^+ y^+)\right] \qquad (7)$$

from $y_l^+ < y^+ < y_i^+$

$$\frac{\varepsilon_m}{\nu} = \frac{K^2 (\mathrm{d}u^+/\mathrm{d}y^+)^3}{(\mathrm{d}^2 u^+/\mathrm{d}y^{+2})^2} \qquad (8)$$

with

$$l = K \frac{(\mathrm{d}u/\mathrm{d}y)}{(\mathrm{d}^2 u/\mathrm{d}y^2)} \qquad (9)$$

where K is an experimentally determined constant.

For $y_i^+ < y^+ < y_0^+$, the expression for ε_m/ν is a complex function of both z and Re. However,

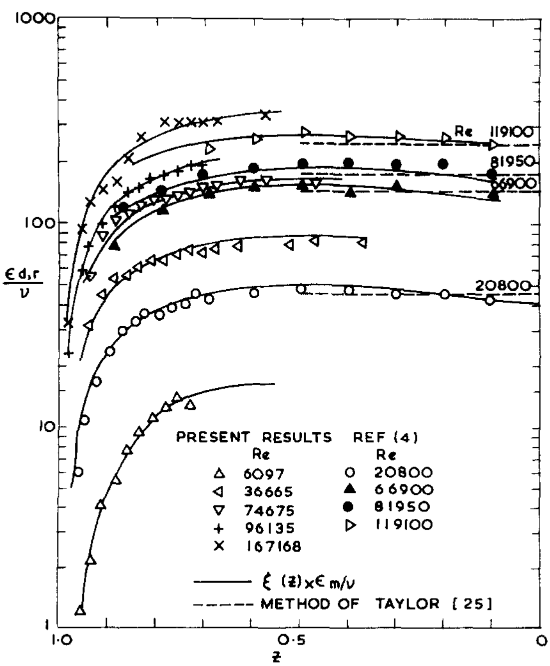


FIG. 10. Radial eddy diffusivity of mass variation with non-dimensional radius.

The $u^+ \sim y^+$ relationship needed to evaluate equation (4) and (5) was obtained from the above description of ε_m by integration of equation (6) and using the results of [15] for the sub-layer thickness y_l^+. As stated, this prediction for $u^+ \sim y^+$ was in excellent agreement with experiment. Together with the fact that the description of ε_m used satisfies most of the theoretical constraints listed by [18], this result is regarded as establishing the superior validity of the description of ε_m used in a situation where the experimental evidence for ε_m is rather inconclusive.

RESULTS AND DISCUSSION

Having established ε_m and u^+ as functions of y^+ or equivalently r^+, $\varepsilon_{h,r}$ and $\varepsilon_{d,r}$ were evaluated from equation (4). With heat transfer the value of $\partial\theta/\partial r^+$ at the wall was not easy to determine. Two methods were used: firstly, by a simple heat balance in which the heat flow at the wall, $-k\ \partial\theta/\partial r^+$, was equated to the change in bulk temperature found by integrating the measured temperature profile and, secondly, by use of the numerical techniques of the appendix. It was considered necessary to use two methods for this particular value of $\partial\theta/\partial r^+$ since the technique of numerical differentiation is less reliable at the end point of the data values and it was not possible to measure the temperature of the wall itself with the same certainty as the temperature of the air stream.

Measurements of $\varepsilon_{d,r}$ as a function of z for some of the Reynolds numbers investigated are

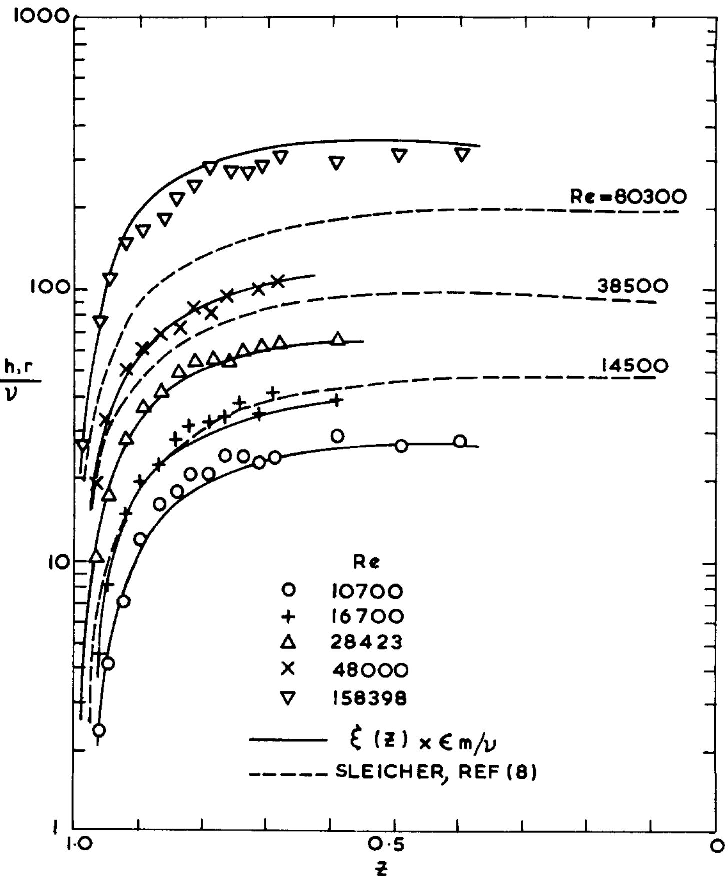


Fig. 11. Radial eddy diffusivity of heat variation with non-dimensional radial.

shown in Fig. 10. It can be seen that the results for $20000 < Re < 30000$ agree with similar results obtained by [4]. In making this comparison the latter were re-evaluated using the better numerical techniques of the appendix. Typical measurements of $\varepsilon_{h,r}$ are shown in Fig. 11. The results of Sleicher [8] are indicated here also.

Using the expression for ε_m given by equations (7)–(10) these results were expressed as the ratio of $\varepsilon_{d,r}$ to ε_m and $\varepsilon_{h,r}$ to ε_m respectively. They are shown in Fig. 12 on which are included the re-evaluated results of [4], also. It is clear that within the limits of experimental error all the results fit a single curve whether they are for heat transfer or mass transfer. It appears from Fig. 12 that the ratio of $\varepsilon_{h,r}$ to ε_m and the ratio of $\varepsilon_{d,r}$ to ε_m are the same. Accordingly they are denoted by the same symbol, ξ, equal to $\varepsilon_{h,r}/\varepsilon_m$ or $\varepsilon_{d,r}/\varepsilon_m$ as appropriate, in the rest of this discussion. It seems that this common ratio, ξ, is a simple function of z and there is no Reynolds number effect. We may use the result for $\xi(z)$, Fig. 12, with ε_m, equations (7)–(10) to predict $\varepsilon_{d,r}$ and $\varepsilon_{h,r}$ and, as Figs. 10 and 11 show, there is very satisfactory agreement between this prediction and the original measurements.

Many theories have attempted to predict this ratio. Some have been based on consideration of the behaviour of individual eddies. Both Jenkins [19] and Azer and Chao [20] chose a model of a spherical eddy exchanging heat as it moved from one point to another. Azer and Chao expressed their results more simply than Jenkins. They found, for $Pr > 0{\cdot}6$.

$$\xi = \frac{1 + 135\,Re^{-0\cdot45}\exp(-z^{0\cdot25})}{1 + 57\,Re^{-0\cdot46}\,Pr^{-0\cdot58}\exp(-z^{0\cdot25})} \quad (11)$$

and for $Pr < 0{\cdot}6$

$$\xi = \frac{1 + 135\,Re^{-0\cdot45}\exp(-z^{0\cdot25})}{1 + 380\,(Re\;Pr)^{-0\cdot58}\exp(-z^{0\cdot25})}. \quad (12)$$

For $Pr > 0{\cdot}6$, Azer and Chao's theory predicts that ξ is greater than unity at the wall. Further, the ratio decreases with Reynolds number for a fixed value of z. The present experimental results support the first of these conclusions but not the second.

Jenkins' results for ξ were expressed in terms of a certain mixing length and a fictitious eddy velocity. To be useable these parameters have to be associated with ε_m and ξ may thus be expressed as a function of ε_m. Jenkins' expression for ξ is thus

$$\xi = Pr\frac{1 - \dfrac{90}{\pi^6}\dfrac{\varepsilon_m Pr}{\nu}\displaystyle\sum_1^\infty \frac{1}{n^6}\left[1 - \exp\left(\frac{-n^2\pi^2\nu}{Pr\,\varepsilon_m}\right)\right]}{1 - \dfrac{90}{\pi^6}\dfrac{\varepsilon_m}{\nu}\displaystyle\sum_1^\infty \frac{1}{n^6}\left[1 - \exp\left(\frac{-n^2\pi^2\nu}{\varepsilon_m}\right)\right]}. \quad (13)$$

Obviously, to find the variation of ξ with Reynolds number and z we need the relationship between ε_m and Reynolds number and z. The one established above equations (7)–(10) was used for this.

Figure 13 compares the prediction of Azer and Chao and Jenkins with the line of best fit for the present measurements. It is clear that neither fits the experimental results well although Azer and Chao seem better to predict the trend that ξ increases towards the wall.

Jenkins considered that his model could not be expected to hold in regions of weak turbulence such as near the wall. In the present results for the lower Reynolds numbers some of the measurements were obtained in such circum-

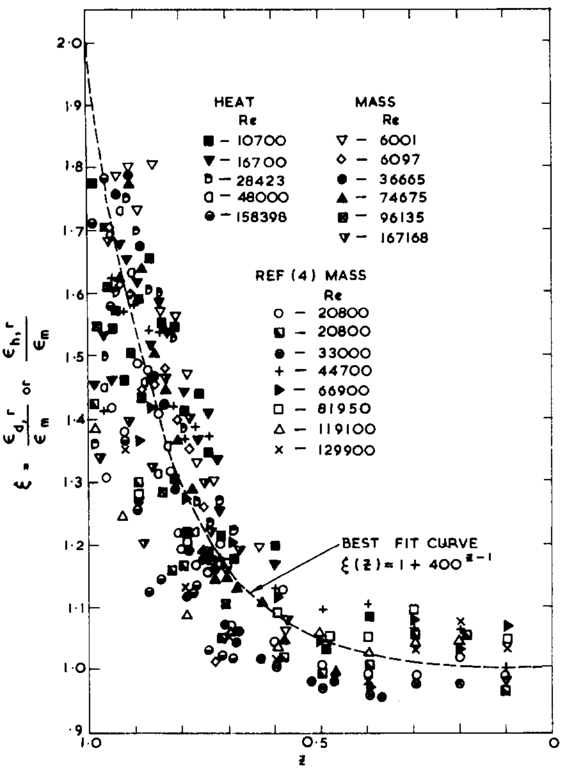


FIG. 12. Ratio of radial mass and heat eddy diffusivities to that of momentum.

stances. For example, with $Re = 6001$ the sublayer, based on the analysis of Quarmby [15], would extend to $z = 0{\cdot}83$. On Jenkins' own arguments we would not expect his prediction to agree well with such results. Marchello and Toor [21] in developing a penetration type model deduced that Jenkins' model would be good within the core but poor elsewhere. Tien [22] also criticised Jenkins' work on the grounds that the model of a solid sphere eddy should only be valid where molecular effects predominate, that is, in liquid metals.

But, some of the present measurements for the higher Reynolds number show quite distinctly an increase in ξ towards the wall. Such values are outside the sublayer. For $Re = 167\,168$ for example the sublayer extends to $z = 0{\cdot}996$. This trend is in agreement with the experimental results of many researchers, for example, those given by Sleicher [8]. Accordingly it is felt that Jenkins' prediction for Prandtl or Schmidt of 0·7, that the ratio decreases towards the wall, is incorrect irrespective of the level of turbulence.

Azer and Chao also used a solid sphere eddy model. The coefficient of heat transfer from the surface of the eddy was taken from laminar boundary layer theory for a flat plate. Azer and Chao's model is subject to the same kind of criticism as the Jenkins' model.

Lykoudis and Toulukian [23] using a spherical eddy model derived the expression

$$\xi = \frac{6}{\pi^2}\left[\exp\left(-\frac{b}{Pr}\right) + 0{\cdot}25\exp\left(-\frac{4b}{Pr}\right)\right] \quad (14)$$

where b is an experimentally determined constant which is independent of both distance from

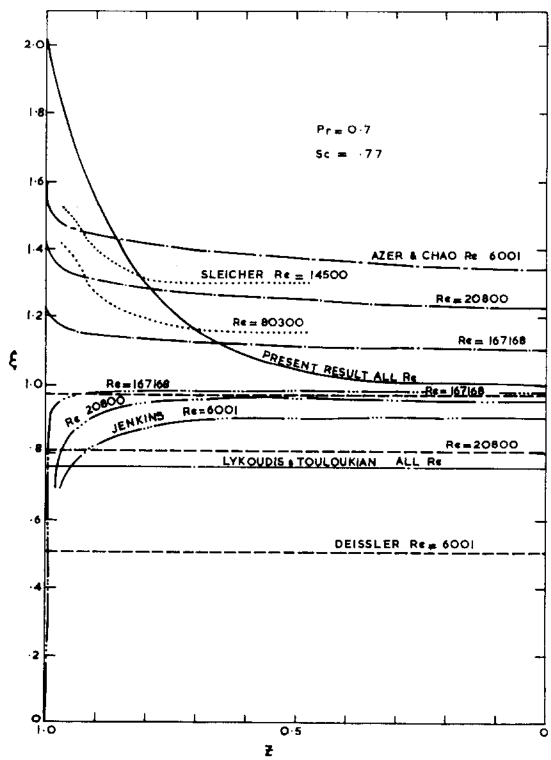


FIG. 13. Ratio of radial mass and heat diffusivities to that of momentum, comparison with various theories.

the wall and Reynolds number. Diessler [24] using a model similar to Jenkins found

$$\xi = n\,Re\,Pr\,[1 - \exp(-n\,Re\,Pr)] \qquad (15)$$

where n is an experimental constant. This expression is independent of wall distance but is a function of Reynolds number. Thus none of the theories mentioned, although using much the same spherical eddy model which exchanges heat or mass in one way or another as it moves about, leads to a result in agreement with the present findings. Clearly, there is room for much improvement in theories of this nature.

The results presented above have been obtained from consideration of equation (3). Taylor [25] has shown how, with certain simplifying assumptions, results may be obtained for the core region. These results do not involve the calculation of $\partial\theta/\partial r^+$ but need the assumption that over a certain region of the core both velocity and eddy diffusivity are constant. Groenhof [26] used Taylor's theory to obtain $\varepsilon_{d,r}$ from his experiments over a range of Reynolds numbers from 25000 to 75000. Several of the central diffusion experiments of [4] were treated in this fashion over the region $0{\cdot}5 > z > 0$. The values for $\varepsilon_{d,r}$ thus obtained are shown in Fig. 10 where it may be seen that they agree well with the values obtained by use of equation (3).

The results for the ratio of the tangential and radial eddy diffusivities of heat and mass are shown in Fig. 14.

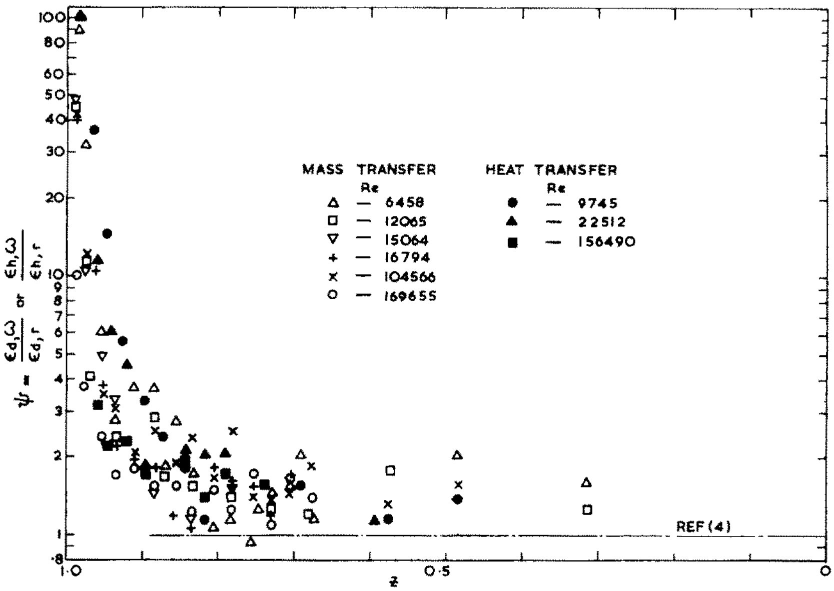


FIG. 14. Ratio tangential mass and heat eddy diffusivities to radial mass and heat eddy diffusivities.

Some measurements of [4] for $\varepsilon_{d,\omega}/\varepsilon_{d,r}$ are compared with the present measurements of $\varepsilon_{d,\omega}/\varepsilon_{d,r}$ in Fig. 14. They found that the ratio was unity but, as can be seen, their results were for the core only. The only previous work for $\varepsilon_{h,\omega}/\varepsilon_{h,r}$ is that of Sparrow and Black [3]. These authors did not give point measurements for $\varepsilon_{h,\omega}/\varepsilon_{h,r}$ but found that they needed to make its value ten in the sublayer to achieve agreement between theory and experiment for the temperature profiles. They concluded that their observations add strong support to a model of the transport process wherein $\varepsilon_{h,\omega}/\varepsilon_{h,r}$ is substantially greater than unity near the wall but is essentially unity at all other points in the flow.

Some measurements of velocity fluctuations near the pipe wall have been made which are relevant to this question. Recently, Kamalesh and Hanratty [27] used an electrochemical technique to measure the circumferential component of the fluctuating velocity gradient near the pipe wall. They concluded that $(\overline{w'^2})^{\frac{1}{2}}/u$ does not extrapolate to zero at the wall as proposed by Laufer [6]. Sherwood *et al.* [28] derived the same value using a tracer particle technique. It seems clearly established that the tangential component, w', is greater than the radial component v' near the wall and the tangential eddy diffusivity is, correspondingly, greater than the radial. From the measured values of the ratio of $\varepsilon_{d,\omega}$ to $\varepsilon_{d,r}$ and of $\varepsilon_{h,\omega}$ to $\varepsilon_{h,r}$ shown in Fig. 14 it can be seen that, within the limits of experimental error, the results fall on a single curve. There is no evidence that the two ratios are different and accordingly a single symbol, ψ, is used to denote them both. Nor is there any evidence of a Reynolds number effect. The line of best fit to the results for the common ratio, ψ, is a simple function of z. It appears to be considerably greater than unity at the wall but to vary smoothly to become unity at the centre. The results of [4] for mass transfer are also shown.

Within the knowledge of the authors, there are no published theories of turbulent exchange, whether using a spherical eddy model or otherwise, which give the ratio of the tangential and radial diffusivities of heat or mass.

CONCLUSIONS

Accurate measurements were obtained of the radial and tangential eddy diffusivities of heat and of mass in a fully developed turbulent flow in a plain tube. A re-examination of the experimental data available for the eddy diffusivity of momentum lead to an expression for the parameter ε_m/r_0^+ which exhibits some effect of Reynolds number. This agrees with more recent theoretical work on the subject.

The ratio of the radial eddy diffusivity of heat, or of mass, to the radial eddy diffusivity of momentum was found to be a simple function of the non-dimensional wall distance. A single curve fits the results for both heat and mass equally well. The ratio has a value about two near the wall and varies smoothly to unity at the centre. There is no Reynolds number effect on this ratio.

The results for the tangential eddy diffusivity of heat and of mass were expressed as a ratio of the corresponding radial eddy diffusivity. A single curve fits both equally well. The ratio is much greater than unity near the wall and appears to be at least ten there. It falls steeply to become unity over the greater part of the flow. There is no effect of Reynolds number on this ratio. The conclusion that the ratio is greater than unity at the wall is supported by limited previous experimental evidence.

None of the available theoretical models of turbulent exchange which were considered correctly predicted the ratio of the radial eddy diffusivities established here. It is felt that the various criticisms which have been made of these models are well founded and there is much room for improvement. It appears not to be the case that the tangential eddy diffusivity equals the radial eddy diffusivity in the region near the wall for either heat or mass exchange. Some attempts should be made to provide a model of turbulent exchange which predicts and explains this observation.

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APPENDIX

Evaluation of First and Second Derivatives of the Temperature and Concentration Profiles

For mass transfer, equation (2) may be transposed to read

$$u^+ r_0^+ \frac{\partial \theta}{\partial x^+} = \frac{1}{z}\frac{\partial}{\partial z}\left[z\left(\frac{1}{Sc} + \frac{\varepsilon_{d,r}}{\nu}\right)\frac{\partial \theta}{\partial z}\right] + \frac{1}{z^2}\frac{\partial}{\partial \omega}\left[\left(\frac{1}{Sc} + \frac{\varepsilon_{d,\omega}}{\nu}\right)\frac{\partial \theta}{\partial \omega}\right] \qquad (A.1)$$

and for the axisymmetric situation this can be written as

$$r_0^+ \int_{z=1}^{z} z u^+ \frac{d\theta}{\partial x^+} dz = \left[z\left(\frac{1}{Sc} + \frac{\varepsilon_{d,r}}{\nu}\right)\frac{d\theta}{dz}\right]_{z=1}^{z} \qquad (A.2)$$

or

$$r_0^+ \int_{z=0}^{z} z u^+ \frac{d\theta}{\partial x^+} dz = \left[z\left(\frac{1}{Sc} + \frac{\varepsilon_{d,r}}{\nu}\right)\frac{d\theta}{dz}\right]_{z=0}^{z}. \qquad (A.3)$$

The following is a consequence of the boundary conditions,

$$r_0^+ \int_{z=1}^{0} z u^+ \frac{d\theta}{dx^+} dz = \int_{z=0}^{1} z u^+ \frac{d\theta}{dx^+} dz = 0 \qquad (A.4)$$

and hence the closure of these integrals is an important clue to the accuracy of the u^+ expression and the derivative $d\theta/dx^+$.

In the experimental determination of temperature and concentration profiles, it was found necessary to take readings at unequal intervals because of a rapid change of the concentration or temperature profile in the region close to the wall. Consequently the usual forward, central and backward difference formulae were found unsuitable.

Various methods of calculating derivatives were tried such as simple linear differentiation, spline fitting, and determining equispaced data by interpolation so that the usual difference formulae were applicable. Lanczos [29] describes a method of differentiation of equispaced data in which he fits a least squares quadratic to five points using a moveable strip technique. The quadratic is then differentiated analytically. This method was extended by the authors to unequispaced data with the following modification. Except at the wall only the central value of the derivative was used, and for axial derivatives, since a large number of points were involved, the fitting of a quadratic to three points only was found suitable. The method was tried successfully on various analytical curves. Moreover the derivatives of the concentration profile met the criterion stated by equation (A.4).

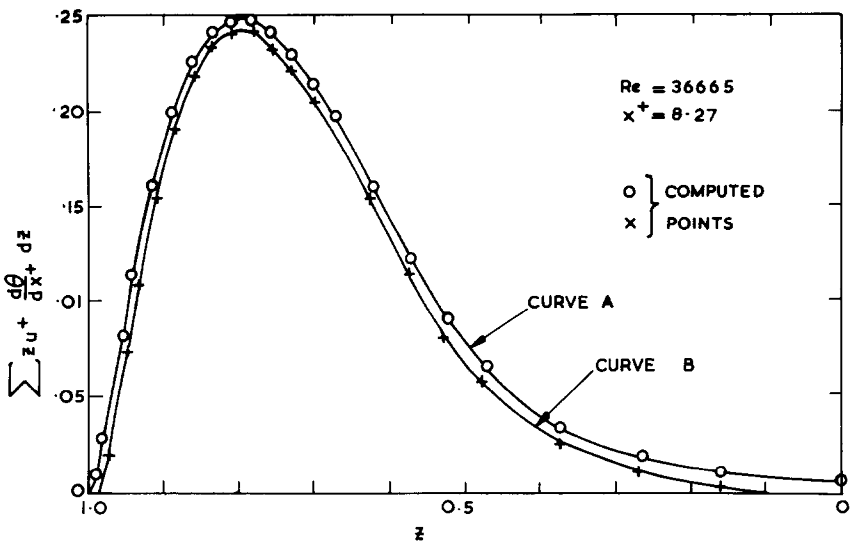

FIG. 15. Axisymmetric mass transfer, illustration of data reduction criterion.

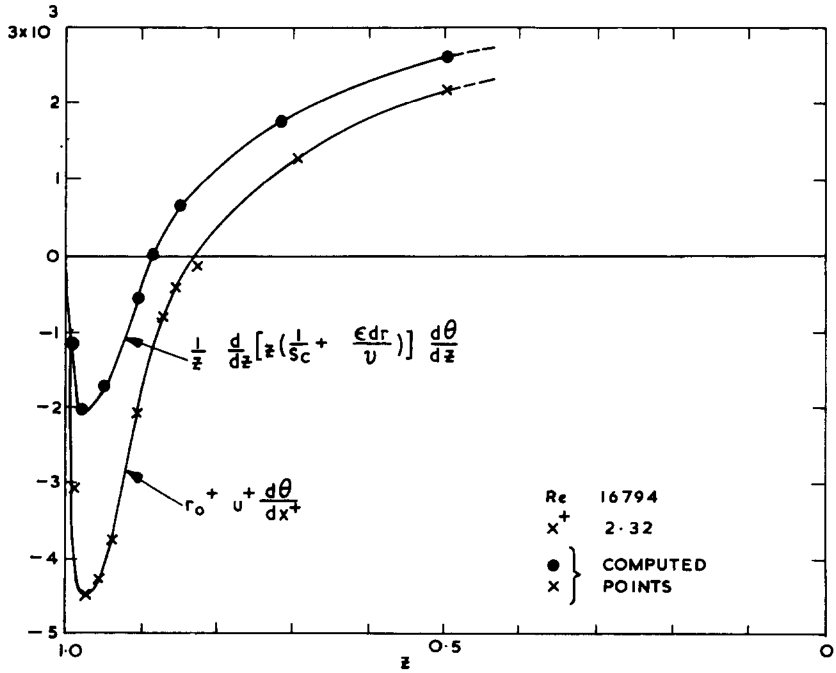

FIG. 16. Non-axisymmetric mass transfer, illustration of data reduction criterion.

Integration techniques consisted of simple trapezoidal or Simpson's rule. A typical plot of the integrals of equation (A.4) is shown in Fig. 15. The integral will naturally show greater error as the integration proceeds over the radius because of the summation of round off and process errors. Thus curve A will be more accurate than curve B close to the wall and vice-versa near the tube centre. This may be seen in the figure.

These methods were used also for axisymmetric heat transfer, except that the integrals do not close for that case.

Equation (A.2), for heat transfer becomes

$$r_0^+ \int_{z=1}^{z} z u^+ \frac{d\theta}{dx^+} dz = \left[z\left(\frac{1}{Pr} + \frac{\varepsilon_{d,r}}{\nu}\right)\frac{d\theta}{dz}\right] - \frac{1}{Pr}\left(\frac{d\theta}{dz}\right)_{z=1}. \tag{A.5}$$

The last term was determined by the method of least squares described above, but also more accurately by differentiating the bulk mean temperature in the axial direction. This can easily be obtained from equation (A.5) by noting that

$$r_0^+ \int_{z=1}^{0} z u^+ \frac{d\theta}{dx^+} dz = -\frac{1}{Pr}\left(\frac{d\theta}{dz}\right)_{z=1}. \tag{A.6}$$

The angular concentration profiles obtained from non-axisymmetric tests were directly differentiable in the angular direction by the least squares method of Lanczos since equispaced data was obtained. However it was found that the method failed to be accurate when the steplength was large, e.g. 15°. Interpolated point values of data could be obtained at a fixed small angular steplength by fitting a Fourier series to the available data and solving for the coefficients by Gaussian Elimination. This new data was then differentiated twice by a double application of the Lanczos moveable strip technique.

Equation (A.1) can be transposed to read:–

$$\left(\frac{\varepsilon_{d,\omega}}{\nu}\right)_z = z^2 \left[u^+ r_0^+ \frac{d\theta}{dx^+} - \frac{1}{z}\frac{d}{dz}\right] \times \left[z\left(\frac{1}{Sc} + \frac{\varepsilon_{d,r}}{\nu}\right)\frac{d\theta}{dz}\right] \Big/ \frac{d^2\theta}{d\omega^2} - \frac{1}{Sc} \tag{A.7}$$

and the derivatives

$$\frac{d\theta}{dx^+}, \frac{d\theta}{dz}$$

and

$$\frac{d}{dz}\left(z\left(\frac{1}{Sc} + \frac{\varepsilon_{d,r}}{\nu}\right)\frac{d\theta}{dz}\right)$$

were determined by the fitting of least squares quadratics as in the axisymmetric case. As a check on the method, graphs of the functions

$$u^+ r_0^+ \frac{d\theta}{dx^+}$$

and

$$\frac{1}{z}\frac{d}{dz}\left[z\left(\frac{1}{Sc} + \frac{\varepsilon_{d,r}}{\nu}\right)\frac{d\theta}{dz}\right]$$

were plotted against z and, since in equation (A.7), $d^2\theta/d\omega^2$ will be of constant sign for a given angular position, the difference between the two graphs must be of constant sign also. A typical plot is shown in Fig. 16 where it can be seen that this criterion is met. The non-symmetric heat transfer situation was found equally amenable to this analysis.

MESURES DES DIFFUSIVITES PAR TURBULENCE RADIALE ET TANGENTIELLE DE CHALEUR ET DE MASSE POUR UN ECOULEMENT TURBULENT DANS UN TUBE

Résumé—Une étude expérimentale a été faite sur la variation des diffusivités par turbulence radiales et tangentielles de chaleur et de masse pour un écoulement turbulent entièrement développé dans un tube. Les résultats concernant la chaleur couvrent le domaine de nombre de Reynolds entre 10 000 et 160 000 et entre 6 000 et 170 000 pour la masse. Dans les deux cas l'air est le fluide porteur à nombre de Prandtl égal à 0,71 avec une injection d'oxyde nitreux à nombre de Schmidt égal à 0,77.

Les résultats pour le transfert massique sont obtenus par diffusion d'oxyde nitreux à partir d'une source annulaire dans la paroi du tube pour la diffusivité radiale par turbulence et par diffusion à partir d'un groupement de sources à la paroi du tube pour la diffusivité tangentielle par turbulence. Les résultats pour le transfert thermique sont obtenus à l'aide d'une source annulaire à la paroi chauffée électriquement ou d'un groupement similaire de sources à la paroi pour respectivement les diffusivités par turbulence radiale et tangentielle.

On a trouvé que les diffusivités par turbulence radiale de chaleur comme de masse peuvent etre exprimées dans un rapport simple de la diffusivité par turbulence radiale de quantité de mouvement pour tous les nombres de Reynolds considérés. Ce rapport est une fonction simple du rayon adimensionnel et il a une valeur proche de deux près de la paroi, mais atteint progressivement l'unité au centre du tube.

De façon similaire on a trouvé que chacune des diffusivités tangentielles par turbulence pouvaient être exprimées comme un rapport simple de la diffusivité radiale par turbulence correspondante; ce rapport est aussi une fonction simple du rayon adimensionnel. Sa valeur est très grande à la paroi mais semble de l'ordre de l'unité dans la majeure partie du tube.

Tous les résultats expérimentaux presentés ont été établis par calculateur. Un appendice traite de la méthode utilisée.

MESSUNG DER RADIALEN UND TANGENTIALEN SCHEINDIFFUSIONSKOEFFIZIENTEN FÜR WÄRME- UND STOFF IN TURBULENTER STRÖMUNG IN EINEM GLATTEN ROHR.

Zusammenfassung—Eine experimentelle Untersuchung wurde durchgeführt über die Veränderung der radialen und tangentialen Scheindiffusionskoeffizienten für Wärme und Stoff in einer voll entwickelten. turbulenten Strömung in einem glatten Rohr. Die Ergebnisse für die Wärmeübertragung liegen im Bereich der Reynoldszahlen von 10000 bis 160000 und die für die Stoffübertragung im Bereich zwischen 6000 und 170000. In beiden Fällen war Luft das Arbeitsmedium mit einer korrespondierenden Prandtl–Zahl von 0,71 und mit Stickstoffoxydul-Einspritzung mit einer Schmidt–Zahl von 0,77.

Die Ergebnisse für die Stoffübertragung wurden mit Stickstoffoxydul gewonnen, das für den radialen Scheindiffusionskoeffizienten aus einer ringförmigen Quelle diffundierte, die glatt in die Rohrwand eingelassen war, und für den tangentialen Scheindiffusionskoeffizienten durch Diffusion aus einer punktförmigen Quelle an der Rohrwand. Die Ergebnisse für die Wärmeübertragung wurden gewonnen unter Verwendung einer elektrisch beheizten Wandringquelle für den radialen, bzw. einer entsprechenden punktförmigen Quelle für den tangentialen Diffusionskoeffizienten.

Es wurde gefunden, dass die radialen Scheindiffusionskoeffizienten für Wärme und Stoff ausgedrückt werden können als einfaches Verhältnis der radialen Scheindiffusionskoeffizienten des Impulses unter Berücksichtigung der Reynoldszahlen. Dieses Verhältnis ist eine einfache Funktion des dimensionslosen Radius und hat einen Wert von ungefähr zwei nahe der Wand aber ändert sich fliessend zu einem einheitlichen Wert in der Rohrachse.

Entsprechend wurde gefunden, dass jeder der tangentialen Schein diffusionskoeffizienen als einfaches Verhältnis seiner korrespondierenden radialen Scheindiffusionskoeffizienten ausgedrückt werden kann. Dieses Verhältnis ist auch eine einfache Funktion von dimensionslosen Radien. Sein Wert an der Wand ist sehr hoch, scheint aber ungefähr einheitlich über den grösseren Teil des Rohres zu sein.

Alle vorgelegten experimentellen Ergebnisse wurden mit Hilfe eines Computers numerisch ausgewertet. Die verwendeten Methoden sind im Anhang zusammengestellt.

ИЗМЕРЕНИЯ РАДИАЛЬНОГО И ТАНГЕНЦИАЛЬНОГО КОЭФФИЦИЕНТОВ ВИХРЕВОЙ ДИФФУЗИИ ТЕПЛА И МАССЫ ПРИ ТУРБУЛЕНТНОМ ТЕЧЕНИИ В ПЛОСКОЙ ТРУБЕ

Аннотация—Проведено экспериментальное исследование изменений радиального и тангенциального коэффициентов вихревой диффузии тепла и массы при полностью развитом турбулентном течении в плоской трубе. Результаты по переносу тепла охватывают диапазон значений критерия Рейнольдса от 10 000 до 160 000, а по диффузии массы- от 6 000 до 170 000. В обоих случаях рабочей жидкостью являлся воздух с соответствующими значениями критерия Прандтля 0,71 и значениями критерия Шмидта 0,77 при вдуве закиси азота.

Результаты по переносу массы были получены при диффузии закиси азота от кольцевого источника, заделанного заподлицо в стенке тьубы для коэффициента вихревой диффузии в радиальном направлении и при диффузии от небольшого плоского источника на стенке трубы для коэффициента вихревой диффузии в тангенциальном направлении. Результаты по переносу тепла были получены с использованием электрически нагреваемого кольцевого источника на стенке или аналогичного плоского источника для радиального и тангенциального коэффициентов вихревой диффузии соответственно.

Найдено, что радиальные коэффициенты вихревой диффузии как тепла, так и массы могут быть выражены как простое отношение радиального коэффициента вихревой диффузии количества движения для всех рассматриваемых значений критерия Рейнольдса. Это отношение является простой функцией безразмерного радиуса и имеет значение, равное примерно двум у стенки, а к центру трубы плавно уменьшается до единицы.

Аналогично было найдено, что каждый из тангенциальных коэффициентов вихревой диффузии может быть выражен как простое отношение их соответствующих коэффициентов радиальной вихревой диффузии; это отношение также является простой функцией безразмерного радиуса. Его значение очень велико на стенке, но равно приблизительно единице в большей части трубы.

Все экспериментальные результаты были обработаны на вычислительной машине. В приложении рассматриваются используемые методы.